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Graded contractions of representations of special linear Lie algebras with respect to their maximal parabolic subalgebras

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Abstract. Parabolic gradings of the classical simple Lie algebras $sl(N, \mathbb{C})$ ($N \geqslant 3$) are described for all maximal parabolic subalgebras. All contractions are found which leave a maximal parabolic subalgebra intact and which preserve a parabolic grading (parabolic contractions of Lie algebras). Contractions of the irreducible representations of $sl(N, \mathbb{C})$ for each parabolic contraction of the Lie algebra are the main results of the article.

1. Introduction

The purpose of the article is to apply the recently developed theory of graded contractions of Lie algebras [1] and their representations [2, 3] to one of the most important classes of cases, namely to simple Lie algebras contracted in such a way that the largest subalgebras (maximal parabolic subalgebras) remain intact. Another equally important class of cases consists of contractions of simple Lie algebras during which maximal reductive subalgebras remain without modification.

In this article we consider only the classical simple Lie algebras of type $sl(N, \mathbb{C})$, mainly to reduce the length of the paper. The cases of orthogonal and symplectic Lie algebras will be considered in a separate article. Here those algebras are sometimes brought up either to underline the similarity/difference of a particular property, or when a body of results applies both to $o(N, \mathbb{C})$ and $sp(2n, \mathbb{C})$. An appealing aspect of our approach is that the problem can be solved simultaneously for all ranks in each series of Lie algebras and simultaneously for all maximal parabolic subalgebras. There are no similar contraction problems considered in the literature in this way.

The standard theory of deformations/contractions of Lie algebras would proceed by studying singular transformations of the third degree tensor of the structure constants. Consequently such a study needs to fix the dimensions of the Lie algebras and has to consider each dimension as a new problem [4].

Much more efficient from a physicists point of view is the method of Gromov [5, 6] where deformation of representations of unitary Cayley–Klein algebras are considered.

The vast number of cases, which our method allows us to study simultaneously, contains many particularly interesting ones for applications in physics and mathematics. Indeed, one may think of the maximal parabolic subalgebras as Lie algebras of inhomogeneous transformations, i.e. the semidirect product of an Abelian ideal, 'translations', with a reductive Lie algebra of 'homogeneous' transformations. The homogeneous part of a maximal parabolic subalgebra of $sl(N, \mathbb{C})$ is a maximal reductive subalgebra. It is well known that in the case of $sl(N, \mathbb{C})$ such a subalgebra decomposes into three ideals (see

(2.1) below). A parabolic contraction then becomes a contraction of $sl(N, \mathbb{C})$ such that a maximal subalgebra of inhomogeneous transformations remains intact. The best known cases of individual parabolic contractions are those of the real Lie algebras of the de Sitter groups to the Lie algebra of the inhomogeneous Lorentz group [7]. This problem is studied in [8]. A general approach to the study of graded contractions of Casmir operators is in [9].

In this article we consider the Lie algebras over the complex field mainly because it is the core of the contraction problem even for the real field, and because the reality conditions can be imposed subsequently in a standard way on the complex case, splitting it into several real ones.

In order also to provide the parabolic contractions for all irreducible representations under consideration, we first study the generic situation. In this way we leave out a small number of irreducible representations of the lowest dimensions. These special cases bring further simplifications but have to be considered separately. Given their importance in applications, we consider them in the subsequent section of the article. The last section deals with the contraction of tensor products.

Let us first recall the main ingredients of our theory [1–3] since it does not resemble the traditional theory of contractions (deformations) of Lie algebras and their representations (see for example the reviews [10–12]).

A simultaneous grading of a chosen Lie algebra L and of its representations is assumed. More precisely there is a subgroup G of the group of automorphisms Aut_L of L whose action on L,

$$gLg^{-1} \qquad g \in G \tag{1.1}$$

is Abelian. In the cases of interest here G is always a cyclic group $\mathbb{Z}_N$ of order N. We chose a primitive element of $\mathbb{Z}_N$, say g, and decompose L into the g-eigenspaces.

$$L_k = \left\{X \in L \mid gXg^{-1} = \mathrm{e}^{2\pi \mathrm{i}k/N}X \quad g \in G\right\}. \tag{1.2}$$

Let the representation $\phi(G)$ act on V. Then we can decompose V into its g-eigenspaces

$$V_m = \left\{v \in V \mid g \in G,\ \phi(g)v = \mathrm{e}^{2\pi \mathrm{i}m/N}v\right\}. \tag{1.3}$$

Thus we have

$$L = \sum_{k (\mathrm{mod}\ N)} L_k \tag{1.4}$$

$$V = \sum_{m (\mathrm{mod}\ N)} V_m \tag{1.5}$$

where G is a finite Abelian (grading) group [3], and the relations

$$[x, y] = z \qquad x \in L_j \quad y \in L_k \quad z \in L_{j+k} \tag{1.6}$$

$$xv = v' \qquad x \in L_j \quad v \in V_m \quad v' \in V_{j+m}\,. \tag{1.7}$$

We assume that not all the relations (1.6) and (1.7) are zero (generic situation). We write for simplicity of notation

$$[L_j, L_k] \subseteq L_{j+k} \tag{1.6$'$}$$

$$L_j V_m \subseteq V_{j+m} \tag{1.7$'$}$$

instead of (1.6) and (1.7) because we intend to consider the contractions which preserve the chosen grading. The contracted commutator $[\ ,\ ]_\varepsilon$ is then defined by the uncontracted one and the parameters ε:

$$[x, y]_\varepsilon = \varepsilon_{jk}[x, y] \qquad x \in L_j \quad y \in L_k\,. \tag{1.8}$$

The requirement that a contraction L^ε is a Lie algebra yields the system of quadratic equations for the contraction parameters [1, 3],

$$\varepsilon_{jk}\varepsilon_{m,j+k} = \varepsilon_{km}\varepsilon_{j,k+m} = \varepsilon_{mj}\varepsilon_{k,m+j} \qquad j, k, m \bmod N \tag{1.9}$$

although in the non-generic cases only a subset of (1.9) may be needed.

The action of L^ε in V is defined by the uncontracted action LV and contraction parameters (ψ_{jm}), $(j, k, m \bmod N)$. More precisely,

$$x \overset{\psi}{\cdot} v = \psi_{jm} x v \qquad x \in L_j \quad v \in V_m\,. \tag{1.10}$$

The result is a representation of L^ε in V provided that one has [2, 3]

$$\varepsilon_{jk}\psi_{j+k,m} = \psi_{km}\psi_{j,k+m} = \psi_{jm}\psi_{k,j+m} \qquad j, k, m \bmod N\,. \tag{1.11}$$

Finally, a tensor product of two representations acting in V and W is a representation of the contracted Lie algebra L^ε with its action in V and W determined by a solution $\psi = (\psi_{jm})$ of (1.11), provided the product is modified as follows:

$$x \overset{\tau}{\otimes} y = \tau_{jk} x \otimes y \qquad x \in V_j \quad y \in W_k \tag{1.12}$$

where the contraction parameters τ_{jk} are subject to the conditions [2, 3]

$$\psi_{jk}\tau_{j+k,m} = \psi_{jm}\tau_{k,j+m} = \psi_{j,k+m}\tau_{k,m} \qquad j, k, m \bmod N\,. \tag{1.13}$$

A grading of a Lie algebra L implies that L as a linear space is decomposed into a direct sum (1.4) of grading subspaces where the summation extends over an Abelian group G. We say that the grading *displays* a subalgebra $L' \subset L$ if L' is a sum of several of the subspaces L_k. In this article we are concerned with maximal parabolic subalgebra of the simple Lie algebra A_n over $\mathbb{C}$ and its *parabolic gradings*. We call a grading parabolic if it is the coarsest grading that displays a maximal parabolic subalgebra. Such grading is unique up to the action of the group of automorphisms of L for each maximal parabolic subalgebra. The number of parabolic gradings of a simple Lie algebra L to consider equals the number of maximal parabolic subalgebras, i.e. it is equal to the rank of L.

The grading group G of a parabolic grading of a classical Lie algebra is a cyclic group of inner automorphisms. Namely, one has

$$G = \begin{cases} \mathbb{Z}_3 & \text{for } A_n, n \geqslant 2 \\ \mathbb{Z}_5 & \text{for } B_n, C_n, n \geqslant 2 \quad D_n, n \geqslant 4 \end{cases} \tag{1.14}$$

with exceptions in D_n and C_n where for some maximal parabolic subalgebra the grading group is $\mathbb{Z}_3$. In the case of exceptional simple Lie algebras one also encounters parabolic gradings $\mathbb{Z}_7, \mathbb{Z}_9, \mathbb{Z}_{11}$ and $\mathbb{Z}_{13}$. The parabolic gradings of $sl(N, \mathbb{C})$ are described in section 2.

In addition to the grading decomposition (1.4), an important characteristic of a grading of L for our purposes is the G-structure matrix $\kappa = (\kappa_{jk})$, $j, k \pmod N$ defined as follows [1]:

$$\kappa_{jk} = \begin{cases} 1 & \text{if } [L_j, L_k] \neq 0 \\ 0 & \text{if } [L_j, L_k] = 0\,. \end{cases} \tag{1.15}$$

Note that by definition $\kappa = \kappa^{\mathrm{T}}$. In studying the parabolic gradings of the classical Lie algebras, one encounters only one G-structure for each grading group; namely,

$$\kappa = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \emptyset & 1 \\ 1 & 1 & \emptyset \end{pmatrix} \qquad \text{for } \mathbb{Z}_3 \tag{1.16}$$

$$\kappa = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \emptyset & 1 & 1 \\ 1 & \emptyset & \emptyset & 1 & 1 \\ 1 & 1 & 1 & \emptyset & \emptyset \\ 1 & 1 & 1 & \emptyset & 1 \end{pmatrix} \qquad \text{for } \mathbb{Z}_5\,. \tag{1.17}$$

In these and subsequent contraction matrices, $\emptyset$ represents a zero matrix element. The rows and columns of κ (and related matrices throughout the paper) are numbered from 0 to $n-1$ for $\mathbb{Z}_n$. The appearance of $\emptyset$ matrix elements in (1.16) and (1.17) is our convention because the parabolic gradings, as we explain subsequently, do not require that the values of those parameters are defined at all. The upper diagonal block 2×2 in (1.16) and 3×3 in (1.17) is the matrix κ of the appropriate maximal parabolic subalgebra of L.

The goal of this article is to study and to describe three related types of contractions. First we determine all contractions of the classical Lie algebras that preserve parabolic gradings and that leave the corresponding maximal parabolic subalgebras intact. Such contractions we call *parabolic contractions* of the Lie algebras.

Individual graded contractions are described by a matrix ε which plays the role of the matrix κ of (1.15) for the contracted Lie algebra and which is a solution of a subset of (1.9).

Parabolic contractions are described here in a rather concise way for all ranks and all maximal parabolic subalgebras. The computation exploits the method of [1]. Typically there are only a few parabolic contractions of L for each maximal parabolic subalgebra $P \subset L$, namely, two for (1.16) and seven for (1.17).

Secondly, we determine for each parabolic contraction of a $sl(N, \mathbb{C})$, all the corresponding (parabolic) contractions of irreducible representations using the method of [2].

Individual contractions of representations refer to a fixed ε, i.e. fixed contraction of the Lie algebra, and are described by a matrix ψ defined similarly to ε and κ and determined as solutions of a subset of (1.11). In the case of the adjoint representation one has $\varepsilon = \psi$.

The third type of contraction that we need to consider is that of the tensor product of representations described by the matrix τ and obtained as a solution of (1.13) for a fixed ψ. Without it a contraction of a tensor product of representations of L is not a representation of the contracted Lie algebra L^{ε} [2, 3].

The complete reducibility of representations of classical Lie algebras is lost in the process of parabolic contraction. The contracted representations are not completely reducible. Nevertheless, the preservation of the grading allows one to retain a considerable insight into the structure of tensor products of contracted representations [2, 3].

Finally let us underline that, within the graded contraction approach, one has two options on how to define the contraction process. The narrow option follows [1, 2], meaning that as a result of a contraction the grading subspaces (i.e. all choices of their elements) which did not commute before a contraction, may commute after, but the opposite process from commuting grading subspaces to non-commuting ones is excluded by our definition of contraction.

The wider definition of a graded contraction, which we prefer to call a graded deformation, does not exclude the transformation from commuting to non-commuting subspaces. In the extreme, one may start with an Abelian algebra, which trivially admits any grading, and deform it into a non-Abelian one. As long as one insists on the preservation of a chosen grading, the process is again governed by (1.9) for the deformation parameters. The latter definition clearly offers much 'wilder' variety of outcomes of deformations.

In this article we adhere to the narrow definition of the contraction process.

2. Parabolic contractions of $sl(N, \mathbb{C})$

Our first problem in this section is to describe the parabolic gradings of $sl(N, \mathbb{C})$. Then we find all the parabolic contractions of $sl(N, \mathbb{C})$ for each such grading.

A maximal parabolic subalgebra P_λ of $sl(N, \mathbb{C})$, $N \geqslant 2$, is defined up to equivalence under the group of inner automorphisms of $sl(N, \mathbb{C})$ by the requirements that it contains a Borel subalgebra B of $sl(N, \mathbb{C})$ and also a maximal reductive subalgebra L_0. The latter amounts to

$$P_\lambda \supset L_0 = U_1 \times sl(\lambda, \mathbb{C}) \times sl(N-\lambda-1, \mathbb{C}) \qquad 1 \leqslant \lambda \leqslant N-1\,. \tag{2.1}$$

Here U_1 is a one-dimensional reductive subalgebra also denoted by gl$(1, \mathbb{C})$; we also adopted the convention that $sl(1, \mathbb{C})$ is a Lie algebra with only the trivial one-dimensional representation.

It is well known that all Borel subalgebras are $sl(N, \mathbb{C})$-conjugate. Two maximal parabolic subalgebras P_λ, $P_{\lambda'}$ of $sl(N, \mathbb{C})$ are not $sl(N, \mathbb{C})$-conjugate precisely if $\lambda \neq \lambda'$. The notation L_0 in (2.1) is justified by the fact that this subalgebra of $sl(N, \mathbb{C})$ turns out to be the L_0 subspace of the parabolic grading decomposition.

The most transparent description of the parabolic gradings of $sl(N, \mathbb{C})$ is to consider the simple Lie algebra $sl(N, \mathbb{C})$, $N \geqslant 3$, as represented by $N \times N$ matrices

$$sl(N, \mathbb{C}) = \{X \mid X \in \mathbb{C}^{N\times N},\ \mathrm{tr}X = 0\} \tag{2.2}$$

to fix a partition (λ, μ) of N

$$\lambda + \mu = N \tag{2.3}$$

and to introduce the corresponding block-matrix structure of X

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.4}$$

Here

$$\begin{aligned} &A \in \mathbb{C}^{\lambda\times\lambda} \qquad B \in \mathbb{C}^{\lambda\times\mu} \qquad C \in \mathbb{C}^{\mu\times\lambda} \qquad D \in \mathbb{C}^{\mu\times\mu} \\ &\mathrm{tr}A + \mathrm{tr}D = 0\,. \end{aligned} \tag{2.5}$$

The maximal parabolic subalgebras P_λ of $sl(N, \mathbb{C})$ are then faithfully represented by the matrices

$$Y = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}. \tag{2.6}$$

Similar lower triangular matrices represent the maximal parabolic subalgebra $P_{N-\lambda}$. Indeed, one has

$$\begin{pmatrix} 0 & I_\mu \\ I_\lambda & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_\lambda \\ I_\mu & 0 \end{pmatrix} = \begin{pmatrix} D & C \\ 0 & A \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix.

It is easy to verify directly that the decomposition

$$\mathrm{sl}(N, \mathbb{C}) = L_0 + L_1 + L_2 \tag{2.7}$$

where

$$L_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \tag{2.8}$$

is a $\mathbb{Z}_3$-grading of $sl(N, \mathbb{C})$:

$$\begin{aligned} &[L_0, L_0] \subset L_0 \qquad [L_1, L_1] = [L_2, L_2] = 0 \\ &[L_0, L_1] = L_1 \qquad [L_0, L_2] = L_2 \qquad [L_1, L_2] = L_0 \end{aligned} \tag{2.9}$$

from which it is evident that (1.16) is the grading structure matrix in this case as long as $N \geqslant 3$. The case $N = 2$ is degenerate [1]. This grading displays the maximal parabolic subalgebra P_λ as

$$P_\lambda = L_0 + L_1 . \tag{2.10}$$

It could not be coarser and still display P_λ. Hence it is a parabolic grading for all $1 \leqslant \lambda \leqslant n$.

In general the parabolic grading decomposition of $sl(N, \mathbb{C})$ is determined according to (1.2) as the eigenspace decomposition of any element $g \in \mathrm{SU}(N)$ of the $\mathrm{SU}(N)$-conjugacy class of elements of the adjoint order 3 denoted by

$$[2 \ \underbrace{0 \ldots 0}_{\lambda-1} \ 1 \ \underbrace{0 \ldots 0}_{\mu-1}] \tag{2.11}$$

in the Kac coordinates [14]. The integers $[s_0, s_1, \ldots, s_n]$ of (2.11) should be thought of as attached to the nodes of an extended Dynkin diagram of the Lie algebra, with subscripts corresponding to the numbering of the nodes, s_0 being at the extension node. The symbol $s = [s_0, s_1, \ldots, s_n]$ specifies a conjugacy class of elements of finite order in the (compact simple simply connected) Lie group and in particular it determines the eigenvalues of any element g of that class in any representation of the Lie algebra. In the case of the N-dimensional irreducible representation used for the explicit grading (2.8), the general theory [14] gives us g as the unique diagonal unitary matrix of the conjugacy class (2.11), namely

$$g = \begin{pmatrix} I_\lambda \mathrm{e}^{2\pi \mathrm{i}\mu/3N} & 0 \\ 0 & I_\mu \mathrm{e}^{-2\pi \mathrm{i}\lambda/3N} \end{pmatrix} \qquad \mu + \lambda = N - 1. \tag{2.12}$$

Clearly g generates the cyclic group $\mathbb{Z}_{3N}$. Using (2.12),

$$g\begin{pmatrix} A & B \\ C & D \end{pmatrix} g^{-1} = \begin{pmatrix} A & e^{2\pi i/3}B \\ e^{-2\pi i/3}C & D \end{pmatrix}. \tag{2.13}$$

We find that only three out of the $3N$ subspaces L_k are non-empty. Thus we get the $\mathbb{Z}_3$-grading (2.8).

In general, a diagonal representative g of a conjugacy class $s = [s_0, s_1, ..., s_n]$ of the group SU(N) acts on a subspace $V(\omega)$ of weight ω by multiplying every vector of $V(\omega)$ by the eigenvalue

$$e^{2\pi i\langle s,\omega\rangle/M} \qquad M = s_0 + s_1 + \cdots + s_n$$

and for $\omega = \sum_{j=1}^{n} b_j\alpha_j$

$$\langle s, \omega\rangle = \sum_{j=1}^{n} b_j s_j.$$

Our final task in this section is to determine the parabolic contractions of $sl(N, \mathbb{C})$ for every P_λ. Then we also have the corresponding parabolic grading of $sl(N, \mathbb{C})$ with κ as in (1.16). By definition of the parabolic contraction, we must preserve the commutation relations $[L_0, L_0]$, $[L_0, L_1]$ and $[L_1, L_1]$ of P_λ. Consequently a result of the parabolic contraction of $sl(N, \mathbb{C})$ is a Lie algebra L^ε with the same $\mathbb{Z}_3$-grading and with the grading structure matrix ε of the form

$$\varepsilon = \begin{pmatrix} 1 & 1 & x \\ 1 & \emptyset & y \\ x & y & \emptyset \end{pmatrix} \tag{2.14}$$

where x and y are indeterminate.

Following the method described in [1], we have to find the non-trivial solutions of a system of quadratic equations for x, y obtained by substitution of ε from (2.14) into (1.6) and by *removal* from that system all equations containing ε_{11} and ε_{22} (the zero matrix elements of (2.14)). Such a system is readily solved even by hand, the non-trivial solutions being

$$\varepsilon = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \emptyset & \cdot \\ 1 & \cdot & \emptyset \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & \cdot \\ 1 & \emptyset & \cdot \\ \cdot & \cdot & \emptyset \end{pmatrix}. \tag{2.15}$$

In the first case we have L^ε as a Lie algebra with a non-trivial Levy decomposition. Its semi-simple part is $[L_0, L_0]$, and its radical is $L_1 + L_2 + U_1$, where U_1 is the centre of L_0.

In the second case the Abelian subalgebra L_2 commutes with P_λ.

Summarizing, there are two parabolic contractions of $sl(N, \mathbb{C})$, $N \geqslant 3$, for every maximal parabolic subalgebra P_λ, $1 \leqslant \lambda \leqslant N-1$, of $sl(N, \mathbb{C})$. The lowest case $N = 2$, is degenerate in that the $\mathbb{Z}_3$-grading structure matrix is

$$\kappa = \begin{pmatrix} \emptyset & 1 & 1 \\ 1 & \emptyset & 1 \\ 1 & 1 & \emptyset \end{pmatrix} \tag{2.16}$$

rather than (1.16). The corresponding contractions are described as an example, (4.20)–(4.22), in [1]. It turns out that there is a 1-parametric continuum of non-isomorphic parabolic contractions of $sl(2, \mathbb{C})$ described by the grading structure matrix

$$\varepsilon = \begin{pmatrix} \emptyset & 1 & p \\ 1 & \emptyset & \cdot \\ p & \cdot & \emptyset \end{pmatrix} \qquad 0 \leqslant p < \infty. \tag{2.17}$$

3. Parabolic gradings of representations of simple Lie algebras

The parabolic contractions of Lie algebras serve as the departure point for the study of parabolic contractions of representations. A prerequisite for the study is the simultaneous parabolic grading of the Lie algebra and of its representations. The result of such a grading, for a chosen irreducible representation $\phi(L)$ acting in V, is the description of the grading subspaces V_m in (1.5) for which (1.7′) holds. In particular, if $\phi(L)$ is the adjoint representation of L, then there is the one-to-one correspondence

$$L_k \longleftrightarrow V_k \qquad \text{for all } k. \tag{3.1}$$

We proceed by considering at the same time all the representations of L with the generic grading action

$$0 \subset L_k V_m \subseteq V_{k+m} \tag{3.2}$$

for all L_k of L and all V_m of V. Here V may be of finite or infinite dimension. Note the strict inclusion of 0 in $L_k V_m$ meaning that we assume that one does not have $L_k V_m = 0$

An added complication is the fact that a given element g of a grading group acting on the Lie algebra $sl(N, \mathbb{C})$ as an automorphism of order 3, may act on different irreducible representation as an automorphism of order $3K$, where K is a divisor of N (see the second example of this section). We have seen this phenomenon in (2.12) and (2.13). Note that the determinant of the Cartan matrix of $sl(N, \mathbb{C})$ is equal to N.

In order to reconcile the third roots of 1 appearing as the eigenvalues of the action (2.13) of the grading group on the Lie algebra and the roots of 1 of order $3N$ determining the eigenspaces of $\phi(g)$ in V, with the grading requirement (1.7′), we proceed as follows. We equivalently relabel L_0, L_1, L_2 as L_0, L_N, and L_{2N} and read the subscripts modulo $3N$.

The grading subspaces V_m are defined by (1.3) as the eigenspaces of the generating element $\phi(g)$ of the appropriate grading group. Let us recall how one finds the eigenvalues of $\phi(g)$ [14]. Starting from the standard weight decomposition of V,

$$V = \sum_{\omega} V(\omega) \tag{3.3}$$

we have to determine the eigenvalue $\exp(2\pi i m/3N)$ of $\phi(g)$ for every weight ω of V. The general theory [14] provides the following answer for the parabolic grading of a simple Lie algebra corresponding to a given value of λ:

$$\phi(g)V(\omega) = e^{2\pi i a_\lambda/3N} V(\omega) \qquad 1 \leqslant \lambda \leqslant n \tag{3.4}$$

where a_λ/N is the coefficient of the simple root α_λ in the expression

$$\omega = \frac{1}{N} \sum_{j=1}^{N-1} a_j \alpha_j \qquad a_j \in \mathbb{Z} \tag{3.5}$$

for a weight ω as a linear combination of simple roots.

The weights ω of V are readily calculated by a standard algorithm, hence they are known for any representation of L acting in V. A grading subspace V_m is then a direct sum of weight subspaces $V(\omega')$, $V(\omega'')$, labelled by weights

$$\omega' = \frac{1}{N} \sum_{j=1}^{N-1} a'_j \alpha_j \qquad \omega'' = \frac{1}{N} \sum_{j=1}^{N-1} a''_j \alpha_j, \ldots \tag{3.6}$$

such that

$$m = a'_\lambda = a''_\lambda \qquad (\mathrm{mod}\ 3N)\,. \tag{3.7}$$

Any two weights of an irreducible representation differ by a linear combination of simple roots with integer coefficients. Hence every irreducible representation space V decomposes into three eigenspaces of $\phi(g)$,

$$V = V_b + V_{b+N} + V_{b+2N} \qquad 0 \leqslant b < N\,. \tag{3.8}$$

The integer b characterizes the congruence class [11] of the representation V.

Let us now consider two $sl(4,\mathbb{C})$ examples involving the irreducible representations of dimension 10 with the highest weight (200). Let us choose the maximal parabolic subalgebra P_2, i.e. $\lambda = 2$, for the first example.

Let us descibe P_2 in terms of 4×4 matrices (2.8), $P_2 = L_0 + L_1$, where

$$L_0 = \begin{pmatrix} a & b & \cdot & \cdot \\ c & d & \cdot & \cdot \\ \cdot & \cdot & e & f \\ \cdot & \cdot & g & h \end{pmatrix} \qquad L_1 = \begin{pmatrix} \cdot & \cdot & m & o \\ \cdot & \cdot & p & q \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{cases} a, b, \ldots, q \in \mathbb{C} \\ a+d+e+h = 0\,. \end{cases} \tag{3.9}$$

A common convenient choice of the generators of root decomposition of $sl(N,\mathbb{C})$ is the following:

$$h_{\alpha_1} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad h_{\alpha_2} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad h_{\alpha_3} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}$$
$$e_{\alpha_1} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_{\alpha_2} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_{\alpha_3} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{3.10}$$

and $e_{-\alpha_i} = (e_{\alpha i})^T,\ i = 1, 2, 3$, and their commutators

$$e_{\alpha_1+\alpha_2} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_{\alpha_2+\alpha_3} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_{\alpha_1+\alpha_2+\alpha_3} = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \text{etc}\,. \tag{3.11}$$

In terms of the root decomposition of $sl(4,\mathbb{C})$, the subspaces are spanned by the following generators:

$$\begin{aligned} L_0 &= \left\{h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, e_{\pm\alpha_1}, e_{\pm\alpha_3}\right\} \\ L_1 &= \left\{e_{\alpha_2}, e_{\alpha_1+\alpha_2}, e_{\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_2+\alpha_3}\right\} \\ L_2 &= \left\{e_{-\alpha_2}, e_{-\alpha_1-\alpha_2}, e_{-\alpha_2-\alpha_3}, e_{-\alpha_1-\alpha_2-\alpha_3}\right\}\,. \end{aligned} \tag{3.12}$$

In order to describe the corresponding $\mathbb{Z}_3$-grading of the representation (200), let us recall that its weights ω are the following linear combinations of simple roots:

$$\begin{aligned} &(200) = \tfrac{3}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 && (\bar{1}01) = -\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3 \\ &(010) = \tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 && (\bar{1}1\bar{1}) = -\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_3 \\ &(\bar{2}20) = -\tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 && (0\bar{2}2) = -\tfrac{1}{2}\alpha_1 - \alpha_2 + \tfrac{1}{2}\alpha_3 \\ &(1\bar{1}1) = \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3 && (0\bar{1}0) = -\tfrac{1}{2}\alpha_1 - \alpha_2 - \tfrac{1}{2}\alpha_3 \\ &(10\bar{1}) = \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_3 && (00\bar{2}) = -\tfrac{1}{2}\alpha_1 - \alpha_2 - \tfrac{3}{2}\alpha_3 \end{aligned} \tag{3.13}$$

each of multiplicity one; the bar denotes a minus sign. The grading eigenvalue (3.4) is determined, for the case $\lambda = 2$, by the coefficient of α_2. Hence we find the grading subspaces labelled by weights with the same coefficient at α_2,

$$\begin{aligned} V_0 &= V(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3) + V(\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_3) \\ V_1 &= V(\tfrac{3}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3) + V(\tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3) \\ V_2 &= V(-\tfrac{1}{2}\alpha_1 - \alpha_2 + \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 - \alpha_2 - \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 - \alpha_2 - \tfrac{3}{2}\alpha_3)\,. \end{aligned} \tag{3.14}$$

Note that $L_1 V_1 = 0$ and $L_2 V_2 = 0$ follow from (3.12) and (3.13) because the weights of V_1 and V_2 cannot be changed by L_1 and L_2 respectively.

Let us now change the example by replacing P_2 by P_1. Then the eigenvalue in (1.3) is a sixth root of 1 because the coefficient of α_1 in (3.13) is half-odd. We have

$$L_0 = \begin{pmatrix} a & \cdot & \cdot & \cdot \\ \cdot & b & c & d \\ \cdot & e & f & g \\ \cdot & h & j & k \end{pmatrix} \qquad L_2 = \begin{pmatrix} \cdot & l & m & o \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{cases} a, b, \ldots, o \in \mathbb{C} \\ a + b + f + k = 0 \end{cases} \tag{3.15}$$

instead of (3.9), and

$$\begin{aligned} L_0 &= \{h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, e_{\pm\alpha_2}, e_{\pm\alpha_3}, e_{\pm(\alpha_2+\alpha_3)}\} \\ L_2 &= \{e_{\alpha_1}, e_{\alpha_1+\alpha_2}, e_{\alpha_1+\alpha_2+\alpha_3}\} \\ L_4 &= \{e_{-\alpha_1}, e_{-\alpha_1-\alpha_2}, e_{-\alpha_1-\alpha_2-\alpha_3}\} \end{aligned} \tag{3.16}$$

is a $\mathbb{Z}_3$-grading of L but the subscripts in L_k are read modulo 6. The subspaces of V are determined by the coefficients of α_1 in (3.13),

$$\begin{aligned} V_1 &= V(\tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3) + V(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3) + V(\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_3) \\ V_3 &= V(\tfrac{3}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3) \\ V_5 &= V(-\tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 - \alpha_2 + \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_3) \\ &\quad + V(-\tfrac{1}{2}\alpha_1 - \alpha_2 - \tfrac{1}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 - \alpha_2 - \tfrac{3}{2}\alpha_3) + V(-\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_3) \end{aligned} \tag{3.17}$$

with the subscripts of V_m read modulo 6. Here L_2 of (3.15) is spanned by e_{α_1}, $e_{\alpha_1+\alpha_2}$, and $e_{\alpha_1+\alpha_2+\alpha_3}$. But the highest weight of V_3 cannot be raised, hence $L_2 V_3 = 0$. Similarly one finds $L_4 V_5 = 0$. Consequently the parabolic grading of $sl(4, \mathbb{C})$ for $\lambda = 1$ (and also $\lambda = 3$) is not generic for this representation.

4. Parabolic contractions of representations of $sl(N, \mathbb{C})$; the generic case

Assuming that a parabolic contraction L^ε of a classical Lie algebra L has been fixed together with the corresponding grading of a representation space V, and that this is the generic action case (3.2), we want to describe the representation $\psi(L^\varepsilon)$ acting in V.

Let us now consider all the parabolic gradings with $\mathbb{Z}_3$-grading group of $sl(N, \mathbb{C})$. More precisely we admit the range of values of λ given in (2.1). Subsequently when we study the orthogonal and symplectic algebras and their parabolic contractions, it will become evident that the present contractions apply also to the following cases

$$\lambda = 1, n-1,\ n \quad \text{in } o(2n, \mathbb{C}) \qquad n \geqslant 4 \quad (\text{three of } n \text{ cases})$$
$$\lambda = n \quad \text{in } sp(2n, \mathbb{C}) \qquad n \geqslant 2 \quad (\text{one of } n \text{ cases}).$$

In any of these cases there are the two parabolic contractions of L given by the matrices ε of (2.15). Choosing one of them at a time, our task is to solve the system of quadratic equations for $\psi = (\psi_{jk})$ obtained from (1.11) by *removal* of equalities which contain ε_{11} and ε_{22}, the 0 matrix elements of (1.16).

Solved directly (1.11) yields the following results:

$$\varepsilon = \begin{pmatrix} 1 & 1 & \cdot \\ 1 & \emptyset & \cdot \\ \cdot & \cdot & \end{pmatrix} : \ \psi = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x & y & z \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & x & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & 1 & 1 \\ \cdot & x & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ \cdot & \cdot & \cdot \end{pmatrix} \tag{4.1a}$$

$$\varepsilon = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \emptyset & \cdot \\ 1 & \cdot & \emptyset \end{pmatrix} : \ \psi = \begin{pmatrix} 1 & 1 & 1 \\ \cdot & x & y \\ \cdot & z & \cdot \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ \cdot & \cdot & x \\ \cdot & y & z \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ \cdot & \cdot & \cdot \\ x & y & z \end{pmatrix},$$
$$\begin{pmatrix} \cdot & 1 & 1 \\ \cdot & x & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & 1 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & x \end{pmatrix}. \tag{4.1b}$$

Here the parameters in ψ can take any value including 0. When non-zero, in most cases they can be transformed into a chosen value (for example, 1) by renormalizing the grading subsapces of V_k.

In addition to the ψ in (4.1) there are other solutions of (1.11) obtained from those of (4.1) by a cyclic permutation of columns in any ψ of (4.1). Different contractions of representations are given by distinct matrices ψ.

Let us consider an example:

$$\varepsilon = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \emptyset & \cdot \\ 1 & \cdot & \emptyset \end{pmatrix} : \quad \psi = \varepsilon. \tag{4.2}$$

Suppose V is graded as in (3.8). In order to simplify notation let us distinguish the subscripts by the constant multiplying N in (3.8) and reading that constant (mod 3). Thus we write

$$V = V_0 + V_1 + V_2 =: \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \tag{4.3}$$

using the symbolic matrix-column for V.

The contracted action of L^{ε} on V is conveniently written as

$$LV = \begin{pmatrix} \psi_{00}L_0 & \psi_{21}L_2 & \psi_{12}L_1 \\ \psi_{10}L_1 & \psi_{01}L_0 & \psi_{22}L_2 \\ \psi_{20}L_2 & \psi_{11}L_1 & \psi_{02}L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}$$

$$= \begin{pmatrix} L_0 & 0 & 0 \\ L_1 & L_0 & 0 \\ L_2 & 0 & L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} L_0V_0 \\ L_1V_0 + L_0V_1 \\ L_2V_0 + L_0V_2 \end{pmatrix} . \tag{4.4}$$

The actual content of V_0, V_1, and V_2 depends on the choice of the irreducible representation one wants to consider. In general a parabolic grading decomposes V as in (3.8). The dimension of V may or may not be finite.

5. Parabolic contractions of representations of $sl(N, \mathbb{C})$: the non-generic cases

Let us now consider the representations of $sl(N, \mathbb{C})$ excluded by (3.2) imposed on the parabolic grading. Thus we now have $0 \subseteq L_k V_m$ and assume that for some values of k and m, L_k annihilates V_m. Let us first determine the irreducible representations of $sl(N, \mathbb{C})$ in which that happens. These cases are summarized in table 1.

Let us fix the value of λ, $1 \leqslant \lambda \leqslant N-1$. The parabolic decomposition (3.8) of V corresponding to λ is determined by the coefficients of α_λ of weights of the weight system of V. Two weight subspaces $V(\omega')$ and $V(\omega'')$ of V, belong to the same grading subspace V_m with m given by (3.7).

Consider any irreducible representation of $sl(N, \mathbb{C})$ with the highest weight $(x_1, x_2, \ldots, x_{N-1})$ (relative to the basis of the fundamental weights). The difference of the highest and the lowest weight $(-x_{N-1}, -x_{N-2}, \ldots, -x_1)$ is then

$$(x_1 + x_{N-1}, x_2 + x_{N-2}, \ldots, x_\lambda + x_{N-\lambda}, \ldots, x_{N-1} + x_1) = \sum_{j=1}^{N-1} A_j \alpha_j$$

$$A_j \in \mathbb{Z}^{>0} \quad \text{for } 1 \leqslant j \leqslant N-1 . \tag{5.1}$$

If $A_\lambda = 1$ then there are only two non-empty subspaces in (3.8) because α_λ in that weight system has only two values of coefficients. This happens precisely for the lowest-dimensional representations $(10\ldots0)$ and $(0\ldots01)$ of $sl(N, \mathbb{C})$. More precisely we have for $(10\ldots0)$

$$V = V_{N-\lambda} + V_{3N-\lambda} \qquad \dim V_{N-\lambda} = \lambda \qquad \dim V_{3N-\lambda} = N - \lambda \tag{5.2}$$

and the highest weight subspace is in $V_{N-\lambda}$ and the lowest weight subspace is in $V_{3N-\lambda}$.

Consequently,

$$L_N V_{N-\lambda} = 0 \qquad L_{2N} V_{3N-\lambda} = 0 \tag{5.3}$$

because $V_{2N-\lambda} = \varnothing$.

For $(0\ldots01)$ we need to change the sign of the subspaces of V_m in (5.2) and (5.3), while keeping λ unchanged.

Table 1. The parabolic gradings of the irreducible representations $(10\ldots0)$, $(20\ldots0)$, and $(010\ldots0)$ of $sl(N,\mathbb{C})$, $n \geqslant 24$. The value of λ labels a parabolic grading. A subscript of V indicates the grading label of the subspace (mod $3N$); its superscript is the dimension; a preceding dot superscript (subscript) indicate the subspace containg the highest (lowest) weight space of the representation. The EFO shown in the top row provide the parabolic gradings of L and V. The position of $\emptyset$ in the 3×3 matrices gives the values of j and m in the annihilations $L_jV_m = 0$ which make the grading non-generic.

EFO	$[2,1,0,\ldots,0]$	$[2,0,\ldots,0,1,0,\ldots,0]$		$[2,0,\ldots,0,1]$
λ	1	$1<\lambda\leqslant[N/2]$	$[N/2]<\lambda<N-1$	$N-1$
$(010\cdots0)$ dim $=\frac{1}{2}N(N-1)$	${}^{\bullet}V_{N-2}^{N-1}+\emptyset+{}_{\bullet}V_{3N-2}^{(N-1)(N-2)/2}$ $\begin{pmatrix}1&\emptyset&1\\\emptyset&\emptyset&1\\1&\emptyset&\emptyset\end{pmatrix}$	$V_{N-2\lambda}^{\lambda(N-\lambda)}+{}^{\bullet}V_{2N-2\lambda}^{\lambda(\lambda-1)/2}+{}_{\bullet}V_{3N-2\lambda}^{(N-\lambda)(N-\lambda-1)/2}$ $\begin{pmatrix}1&1&1\\1&\emptyset&1\\1&1&\emptyset\end{pmatrix}$	${}^{\bullet}V_{2N-2\lambda}^{\lambda(\lambda-1)/2}+{}_{\bullet}V_{3N-2\lambda}^{(N-\lambda)(N-\lambda-1)/2}+V_{N-2\lambda}^{\lambda(N-\lambda)}$ $\begin{pmatrix}1&1&1\\\emptyset&1&1\\1&\emptyset&1\end{pmatrix}$	${}^{\bullet}V_{2}^{(N-1)(N-2)/2}+\emptyset+{}_{\bullet}V_{2N+2}^{N-1}$ $\begin{pmatrix}1&\emptyset&1\\\emptyset&\emptyset&1\\1&\emptyset&\emptyset\end{pmatrix}$
$(200\ldots0)$ dim $=\frac{1}{2}N(N+1)$	$V_{N-2\lambda}^{\lambda(N-\lambda)}+{}^{\bullet}V_{2N-2\lambda}^{\lambda(\lambda+1)/2}+{}_{\bullet}V_{3N-2\lambda}^{(N-\lambda)(N-\lambda+1)/2}$ $\begin{pmatrix}1&1&1\\1&\emptyset&1\\1&1&\emptyset\end{pmatrix}$		${}^{\bullet}V_{2N-2\lambda}^{\lambda(\lambda+1)/2}+{}_{\bullet}V_{3N-2\lambda}^{(N-\lambda)(N-\lambda+1)/2}+V_{N-2\lambda}^{\lambda(N-\lambda)}$ $\begin{pmatrix}1&1&1\\\emptyset&1&1\\1&\emptyset&1\end{pmatrix}$	
$(10\ldots0)$ dim $=N$	${}^{\bullet}V_{N-\lambda}^{\lambda}+\emptyset+{}_{\bullet}V_{3N-\lambda}^{N-\lambda}$ $\begin{pmatrix}1&\emptyset&1\\\emptyset&\emptyset&1\\1&\emptyset&\emptyset\end{pmatrix}$			

The case $A_\lambda = 2$ occurs for the following irreducible representations

$$(200\ldots00) \quad \text{and} \quad (0\ldots002) \tag{5.4a}$$

$$(010\ldots00) \quad \text{and} \quad (0\ldots010) \tag{5.4b}$$

$$(100\ldots01). \tag{5.4c}$$

For these representations the annihilations $L_k V_m = 0$ are described in table 1.

The grading decomposition of the representation space $(20\ldots0)$ of dimension $\frac{1}{2}N(N+1)$ is as follows:

$$V = V_{N-2\lambda} + V_{2N-2\lambda} + V_{3N-2\lambda} \qquad 1 \leqslant \lambda \leqslant N-1 \tag{5.5}$$

and

$$L_N V_{2N-2\lambda} = L_{2N} V_{3N-2\lambda} = 0 \tag{5.6}$$

where

$$\dim V_{N-2\lambda} = \lambda(N-\lambda) \qquad \dim V_{2N-2\lambda} = \tfrac{1}{2}\lambda(\lambda+1) \qquad \dim V_{3N-2\lambda} = \tfrac{1}{2}(N-\lambda)(N-\lambda+1)\,. \tag{5.7}$$

The decomposition of V of $(0\ldots02)$ is again obtained by reversing the sign of the subspaces of V_m and referring to the same λ.

The representation space of $(010\ldots0)$ for the parabolic gradings $\lambda = 1$ and $N-1$ decomposes like $(10\ldots0)$, and for $1 < \lambda < N-1$ it decomposes like that of $(20\ldots0)$. More precisely

$$V = V_{N-2\lambda} + V_{2N-2\lambda} + V_{3N-2\lambda} \tag{5.8}$$

and

$$\dim V_{N-2\lambda} = \lambda(N-\lambda) \qquad \dim V_{2N-2\lambda} = \tfrac{1}{2}\lambda(\lambda-1) \qquad \dim V_{3N-2\lambda} = \tfrac{1}{2}(N-\lambda)(N-\lambda-1)\,. \tag{5.9}$$

Note that $V_{2N-2\lambda} = \varnothing$ for $\lambda = 1$ and $V_{3N-2\lambda} = \varnothing$ for $\lambda = N-1$.

The grading decompositions of V of $(0\ldots010)$ are obtained again by reversing the signs of subspaces of V_m above.

The adjoint representation $(10\ldots01)$ is graded like the Lie algebra

$$V = V_0 + V_N + V_{2N} \qquad \dim V_0 = \lambda^2 + (N-\lambda)^2 - 1 \qquad \dim V_N = \dim V_{2N} = \lambda(N-\lambda)$$

and $L_N V_N = L_{2N} V_{2N} = 0$.

In table 1 we show the matrix ψ before contraction for all the non-generic cases, i.e. when some $L_k V_m = 0$. They are indicted by the zeros of the matrix ψ.

The representation contractions are now solutions of the subsystem of (1.11) obtained from (1.11) by removal of equations which involve:

(1) ε_{11} and ε_{22}, due to the particularity of parabolic gradings of $\mathrm{sl}(N, \mathbb{C})$ (cf (1.16)), and
(2) equations containg $\psi_{km} = \emptyset$ from table 1.

Solutions of a subsystem can be found directly. It turns out that those solutions are also found among the generic ones (4.1). These are precisely those solutions of (4.1) which have all the zeros required by (2) above.

6. Tensor product contractions

One of the most frequent operations in representation theory is tensor multiplication. The complete reducibility of representations of simple Lie algebras is lost during the contractions. Nevertheless the preservation of a grading (parabolic grading in our case) allows one a considerable insight into the structure of the product [2, 3].

Let V^{ψ} and W^{ψ} be two representation spaces of a contraction L^{ε} of simple Lie algebra L, the action of L^{ε} on V^{ψ} and W^{ψ} being described by ψ. From the simultaneous grading we have

$$V^{\psi} = \bigoplus_{m} V^{\psi}_{m} \qquad W^{\psi} = \bigoplus_{k} W^{\psi}_{k} . \tag{6.1}$$

In general after the contraction we have

$$L^{\varepsilon}\left(V^{\psi} \otimes W^{\psi}\right) \neq L^{\varepsilon} V^{\psi} \otimes W^{\psi} + V^{\psi} \otimes L^{\varepsilon} W^{\psi} . \tag{6.2}$$

In order to transform (6.2) into an equality, as needed if the tensor product of representations should be a representation, we have to introduce a contraction τ of the tensor product [2]

$$(V^{\psi} \otimes W^{\psi})_{\tau} = \left(\bigoplus_{p,q} V^{\psi}_{p} \otimes W^{\psi}_{q}\right)_{\tau} = \bigoplus_{p,q} \tau_{pq} V^{\psi}_{p} \otimes W^{\psi}_{q} . \tag{6.3}$$

Then using τ in (6.2) (see [3] for details), we arrive at the conditions (1.13), for a given ψ.

As in the case of ε and ψ, one can renormalize the grading subspaces in the tensor product problem and thus bring the non-zero matrix elements to a desired value, normally one.

Unlike the equations for ε and ψ, the equations for τ, given a ψ, are linear in τ. Hence a sum of several τ's referring to the same ψ is again a solution of (1.13).

The non-trivial contractions τ for the parabolic $\mathbb{Z}_3$-graded cases of the generic type are found in table 2. Let us consider an example choosing the non-generic case

$$(10\ldots0) \otimes (10\ldots0) = (20\ldots0) \oplus (010\ldots0) . \tag{6.4}$$

The grading decompositions for the three irreducible representations involved here are shown in table 1 as well as the dimensions of their grading subspaces. Putting $V = (10\ldots0)$, $W' = (20\ldots0)$, $W'' = (010\ldots0)$, we have

$$\begin{aligned} &V = V_{N-\lambda} \oplus V_{3N-\lambda} \qquad W' = W'_{N-2\lambda} \oplus W'_{2N-2\lambda} \oplus W'_{3N-2\lambda} \\ &W'' = W''_{N-2\lambda} \oplus W''_{2N-2\lambda} \oplus W''_{3N-2\lambda} . \end{aligned} \tag{6.5}$$

The grading decomposition of the product $V \otimes V$ is thus

$$\begin{aligned} (V \otimes V)_{N-2\lambda} &= V_{N-\lambda} \otimes V_{3N-\lambda} \oplus V_{3N-\lambda} \otimes V_{N-\lambda} = W'_{N-2\lambda} \oplus W''_{N-2\lambda} \\ (V \otimes V)_{2N-2\lambda} &= V_{N-\lambda} \otimes V_{N-\lambda} = W'_{2N-2\lambda} \oplus W''_{2N-2\lambda} \\ (V \otimes V)_{3N-2\lambda} &= V_{3N-\lambda} \otimes V_{3N-\lambda} = W'_{3N-2\lambda} \oplus W''_{3N-2\lambda} \end{aligned} \tag{6.6}$$

Table 2. Solutions τ of (1.13) containing the tensor product contraction parameters τ for given ψ. Conventions: $p \neq q$ and $p, q = 0, 1$; $x \neq y \neq z$ and $x, y, x \in \mathbb{R} \neq 0$; $s, t \in \mathbb{R} \neq 0$; $a, b, c, \ldots, i$ and $w \in \mathbb{R}$.

ψ	τ	ψ	τ	ψ	τ
$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\\cdot&\cdot&x\end{pmatrix}$	$\begin{pmatrix}b&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&a\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\\cdot&x&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&c&b\\\cdot&a&\cdot\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}d&c&d\\b&a&\cdot\\d&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&\cdot&x\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&c\\\cdot&b&a\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&x&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}b&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\x&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}d&d&c\\d&\cdot&\cdot\\b&\cdot&a\end{pmatrix}$
$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\\cdot&x&x\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&a\\\cdot&a&a\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&x&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&\cdot&x\end{pmatrix}$	$\begin{pmatrix}a&a&a\\a&\cdot&\cdot\\a&\cdot&a\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&\cdot&w\\\cdot&x&x\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&a\\\cdot&a&a\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&w&\cdot\\x&x&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\w&\cdot&\cdot\\x&\cdot&x\end{pmatrix}$	$\begin{pmatrix}a&a&a\\a&\cdot&\cdot\\a&\cdot&a\end{pmatrix}$
$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\\cdot&x&y\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&x\\\cdot&x&y\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&y&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\y&\cdot&x\end{pmatrix}$	$\begin{pmatrix}y&y&y\\y&\cdot&\cdot\\y&\cdot&x\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&\cdot&w\\\cdot&x&y\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&x\\\cdot&x&y\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&w&\cdot\\x&y&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\w&\cdot&\cdot\\y&\cdot&x\end{pmatrix}$	$\begin{pmatrix}y&y&y\\y&\cdot&\cdot\\y&\cdot&x\end{pmatrix}$
$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&x&y\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&y&x\end{pmatrix}$	$\begin{pmatrix}x&x&x\\x&y&y\\x&y&x\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\y&x&x\end{pmatrix}$	$\begin{pmatrix}y&y&y\\y&y&x\\y&x&x\end{pmatrix}$
$\begin{pmatrix}1&1&1\\x&x&y\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}x&x&x\\x&x&y\\x&y&y\end{pmatrix}$	$\begin{pmatrix}1&1&1\\x&y&x\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\y&x&x\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}y&y&y\\y&x&x\\y&x&y\end{pmatrix}$
$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&y&z\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\y&z&x\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&x&y\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\x&y&z\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\y&z&x\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\z&x&y\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}p&q&q\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}c&\cdot&\cdot\\\cdot&b&\cdot\\\cdot&\cdot&a\end{pmatrix}$	$\begin{pmatrix}q&q&p\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}c&b&\cdot\\a&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}q&p&q\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}c&\cdot&b\\\cdot&\cdot&\cdot\\a&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&x&x\\\cdot&w&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&a\\\cdot&a&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\x&x&\cdot\\w&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}a&a&a\\a&a&\cdot\\a&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\x&\cdot&x\\\cdot&\cdot&w\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&a\end{pmatrix}$
$\begin{pmatrix}p&q&q\\\cdot&\cdot&\cdot\\\cdot&\cdot&x\end{pmatrix}$	$\begin{pmatrix}b&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&a\end{pmatrix}$	$\begin{pmatrix}q&q&p\\\cdot&\cdot&\cdot\\\cdot&x&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}q&p&q\\\cdot&\cdot&\cdot\\x&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}a&\cdot&a\\\cdot&\cdot&\cdot\\a&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&x&y\\\cdot&w&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&x&y\\\cdot&y&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\x&y&\cdot\\w&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}x&x&x\\x&y&\cdot\\x&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\y&\cdot&x\\\cdot&\cdot&w\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&a\end{pmatrix}$
$\begin{pmatrix}\cdot&1&1\\\cdot&x&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}b&\cdot&\cdot\\x&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&\cdot\\x&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}a&a&\cdot\\a&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&\cdot&1\\\cdot&\cdot&x\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&\cdot&s\\\cdot&\cdot&t\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&a\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&s&\cdot\\\cdot&t&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\s&\cdot&\cdot\\t&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}a&a&a\\a&\cdot&\cdot\\a&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\x&x&x\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}a&a&a\\a&a&a\\a&a&a\end{pmatrix}$	$\begin{pmatrix}p&p&p\\\cdot&\cdot&\cdot\\x&x&x\end{pmatrix}$	$\begin{pmatrix}a&a&a\\a&a&a\\a&a&a\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}i&h&e\\g&f&d\\c&b&a\end{pmatrix}$
$\begin{pmatrix}1&1&1\\\cdot&\cdot&s\\\cdot&t&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&b\\\cdot&a&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&s&\cdot\\t&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&a&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\s&\cdot&\cdot\\\cdot&\cdot&t\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&a\end{pmatrix}$

because the subscripts of the product factors are added (mod $3N$).

Let us now choose L^{ε} given by $\varepsilon = \begin{pmatrix}1&1&1\\1&\emptyset&\cdot\\1&\cdot&\emptyset\end{pmatrix}$ of (2.15) and $\psi = \varepsilon$ as used in (4.4). One of the solutions of (1.13) is always $\tau = \varepsilon = \psi$. Then simplifying again the subscripts in (6.6) by writing it as the coefficient of N (mod 3), we can write

$$
\begin{aligned}
(V \otimes V)_\tau &=: \begin{pmatrix} \tau_{00}V_0 & \tau_{21}V_2 & \tau_{12}V_1 \\ \tau_{10}V_1 & \tau_{01}V_0 & \tau_{22}V_2 \\ \tau_{20}V_2 & \tau_{11}V_1 & \tau_{02}V_0 \end{pmatrix} \otimes \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \begin{pmatrix} V_0 & 0 & 0 \\ V_1 & V_0 & 0 \\ V_2 & 0 & V_0 \end{pmatrix} \otimes \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_0 \otimes V_0 \\ V_1 \otimes V_0 \oplus V_0 \otimes V_1 \\ V_2 \otimes V_0 \oplus V_0 \otimes V_2 \end{pmatrix}.
\end{aligned}
\tag{6.7}
$$

In order to verify that the contracted tensor product is a representation of L^ε with the contracted action $L^\varepsilon(\cdot V)_\psi$ on the individual subspace we have to verify the equality

$$
(L^\varepsilon(V \otimes V)_\tau)^\psi = (L^\varepsilon V \otimes V)^\psi_\tau + (V \otimes L^\varepsilon V)^\psi_\tau . \tag{6.8}
$$

We have the left side using L from (4.4) and $V \oplus V$ from

$$
\begin{aligned}
(L^\varepsilon(V \otimes V)_\tau)_\psi &= \begin{pmatrix} L_0 & 0 & 0 \\ L_1 & L_0 & 0 \\ L_2 & 0 & L_0 \end{pmatrix} \begin{pmatrix} V_0 \otimes V_0 \\ V_1 \otimes V_0 + V_0 \otimes V_1 \\ V_2 \otimes V_0 + V_0 \otimes V_2 \end{pmatrix} \\
&= \begin{pmatrix} L_0(V_0 \otimes V_0) \\ L_1(V_0 \otimes V_0) + L_0\{V_1 \otimes V_0 + V_0 \otimes V_1\} \\ L_2(V_0 \otimes V_0) + L_0\{V_2 \otimes V_0 + V_0 \otimes V_2\} \end{pmatrix}
\end{aligned}
\tag{6.9}
$$

where all the transformations which are left are as before the contraction.

In order to write the right side of (6.8), note that without loss of generality we can do it as the sum of the following two expressions:

$$
\begin{aligned}
((L^\varepsilon V)_\psi \otimes V)_\tau &= \begin{pmatrix} L_0 & 0 & 0 \\ L_1 & L_0 & 0 \\ L_2 & 0 & L_0 \end{pmatrix} \begin{pmatrix} V_0 & 0 & 0 \\ V_1 & V_0 & 0 \\ V_2 & 0 & V_0 \end{pmatrix} \otimes \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \begin{pmatrix} L_0V_0 & 0 & 0 \\ L_1V_0 + L_0V_1 & L_0V_0 & 0 \\ L_2V_0 + L_0V_2 & 0 & L_0V_0 \end{pmatrix} \otimes \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \begin{pmatrix} L_0V_0 \otimes V_0 \\ (L_1V_0 + L_0V_1) \otimes V_0 + L_0V_0 \otimes V_1 \\ (L_2V_0 + L_0V_2) \otimes V_0 + L_0V_0 \otimes V_2 \end{pmatrix}
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
(V \otimes (L^\varepsilon V)_\psi)_\tau &= \begin{pmatrix} V_0 & 0 & 0 \\ V_1 & V_0 & 0 \\ V_2 & 0 & V_0 \end{pmatrix} \otimes \begin{pmatrix} L_0 & 0 & 0 \\ L_1 & L_0 & 0 \\ L_2 & 0 & L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \begin{pmatrix} V_0 & 0 & 0 \\ V_1 & V_0 & 0 \\ V_2 & 0 & V_0 \end{pmatrix} \otimes \begin{pmatrix} L_0V_0 \\ L_1V_0 + L_0V_1 \\ L_2V_0 + L_0V_2 \end{pmatrix} \\
&= \begin{pmatrix} V_0 \otimes L_0V_0 \\ V_1 \otimes L_0V_0 + V_0 \otimes (L_1V_0 + L_0V_1) \\ V_2 \otimes L_0V_0 + V_0 \otimes (L_2V_0 + L_0V_2 \end{pmatrix}.
\end{aligned}
\tag{6.11}
$$

Substitution of (6.9), (6.10), and (6.11) into (6.8) clearly makes both sides equal. All operations which remain are those before the contraction. Hence we have demonstrated that the contracted tensor product $(V \otimes V)_\tau$ is a representation of L^ε. Moreover, it has been achieved by simply striking out certain terms of the uncontracted transformations of $V_1 \otimes V_2$ and $V_2 \otimes V_1$ as well as $V_1 \otimes V_1$ and $V_2 \otimes V_2$ by L.

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